Differentially private consensus of multi-agent systems under binary-valued communications

Xiaodong Lu, Yanlong Zhao, *Senior Member IEEE*, Ji-Feng Zhang, *Fellow IEEE*

***Abstract*— This paper investigates the differentially private output consensus problem of continuous-time heterogeneous multi-agent systems. An algorithm blending estimation, control and privacy is proposed. To prevent privacy leakage, each agent constructs an auxiliary variable regarding to its own initial output, and then adds a non-decaying noise to this variable. To incorporate the network bandwidth limitation, each agent quantifies the noisy auxiliary variable to 1-bit binary-valued data, and transmits this data to its neighbours at some impulse time instants. Based on the received information, each agent builds the recursive projection algorithm to estimate its neighbours' auxiliary variables, and updates its own auxiliary variable. Then, each agent designs its controller by using its own state and auxiliary variable to achieve the mean-square output consensus. To characterize the degree of privacy protection, the differential privacy index of the proposed mechanism is derived. Compared with the existing works, this paper only needs 1-bit communication bandwidth, and the control signals are not required to constantly feed back to the original system. From the aspect of quantization and control, the communication resources can be conserved. An example is presented to illustrate the effectiveness of the theoretical results.**

***Index Terms*— Differentially private consensus, heterogeneous multi-agent systems, binary-valued communications, recursive projection algorithm, impulsive control.**

I. Introduction

Consensus of multi-agent systems (MASs) represents a state where multiple autonomous agents, each with their own objectives, capabilities, and perspectives, come to an agreement or alignment on a particular aspect of their collective behavior or decision. In many practical fields, consensus algorithms are widely applied, including in the power industry [1], [2], multi-spacecraft systems [3], and multi-vehicle coordination [4], etc. In reality, to prevent external eavesdroppers from stealing real communication information, it is imperative to factor in the necessity of privacy protection while striving for consensus.

Differential privacy is one of the main methods of privacy protection, and can precisely characterize the degree of protection by properly choosing some parameters [5], [6]. Another common approach to privacy protection is cryptography, which involves transforming data into an encrypted form and can only be deciphered by the receivers possessing the correct key. Cryptography-based methods can achieve perfect accuracy but often require more computational resources. In contrast, differential privacy offers the advantage of low computational cost, but at the expense of reduced accuracy. In recent years, differentially private consensus has received widespread attention, and some efficient algorithms have been proposed. For example, [7] proposed an algorithm concerning differential privacy and resilient consensus for MASs by adding exponentially decaying privacy noises. [8], [9] proposed differentially private consensus algorithms for continuous-time and discrete-time MASs by adding non-decaying privacy noises with time-invariant variances, respectively. Furthermore, [10] developed a differentially private bipartite consensus algorithm by adding non-decaying privacy noises with time-varying variances. However, all the works [7]–[10] require infinite-bit communication data, which is impractical in real network. Consequently, it is necessary to consider the case with finite-bit bandwidth and quantized communication data.

Many quantized communication techniques can be employed for conserving communication resources, such as infinite-level quantization [11], [12] and finite-level quantization [13]–[15]. Recently, differentially private consensus under quantized communications has been one of active areas, and some important works have been published [16], [17]. In [16], [17], the differentially private consensus problems of first-order and second-order homogeneous MASs with quantized communications were addressed, respectively. In these two works, the transmitted data is generated by the multi-level uniform quantizer. And the privacy noises are required to be exponentially decaying, which may cause privacy leakage as time goes on. Thus, it is necessary to investigate differentially private consensus with non-decaying privacy noises under quantized communications. Except for quantized communications, low-frequency updating control strategies can be adopted to conserve communication resources. These strategies in-

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X. Lu is with School of Automation and Electrical Engineering, University of Science and Technology Beijing, and Key Laboratory of Knowledge Automation for Industrial Processes, Ministry of Education, Beijing 100083, China (e-mail: luxiaodong@ustb.edu.cn).

Y. Zhao is with State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and also with School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China (e-mail: ylzhao@amss.ac.cn).

J. Zhang is with School of Automation and Electrical Engineering, Zhongyuan University of Technology, Zhengzhou 450007, China, and also with State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China (e-mail: jif@iss.ac.cn).

clude, but are not limited to, event-triggered control [18], [19] and impulsive control [20], [21], etc. Additionally, low-frequency communication schemes are also good alternative options, such as event-triggered communication [22], [23] and impulsive communication [24], [25], etc.

Based on the above discussions, this paper investigates the differentially private consensus problem of heterogeneous MASs under binary-valued communications. The main contributions are summarized as follows.

- A communication scheme is established on binary-valued information, which is only 1-bit. Thus, the communication resources can be saved by the scheme of this paper in comparison to that in [8]. The communication process in [8] does not consider the effect of bandwidth restriction and requires infinite-bit transmitted information, which can cause substantial power consumption. It is true that binary-valued information is rather imprecise, but each agent can design the recursive projection algorithm to estimate its neighbours' auxiliary variables.
- The privacy noise employed in this paper can be non-decaying. Consequently, in the sense of privacy protection, the privacy mechanism in this paper is superior to those in [16], [17]. The privacy noises in [16], [17] are required to be exponentially decaying, which is conservative and may result in privacy leakage as time goes on. Indeed, the non-decaying noise can lead to bad consensus and estimation effects. To deal with this issue, this paper utilizes the stochastic approximation approach, in which the decaying stepsize can mitigate the adverse influence of this kind of noise.
- An impulsive control protocol is designed to address the differentially private consensus problem, by which the control signals are generated and affect the original system only at some impulsive instants. As a result, the communication resources can be conserved by the control protocol of this paper compared with that in [8]. The state-feedback control strategy adopted in [8] requires that the control signals constantly feed back to the original system, which may result in resource waste. Despite that the constructed impulsive controllers work only at some impulsive instants, both the consensus error and the estimation error can be guaranteed to asymptotically converge to zero. In addition, [8] only obtains the convergence result of consensus error. Besides, this paper also derives an explicit convergence rate.

The remainder of this paper is organized as follows. Section 2 formulates the considered problem. Section 3 investigates the convergence and the convergence rates of the consensus error and the estimation error. Section 4 obtains the privacy index of the randomized mechanism. Section 5 presents a simulation example to demonstrate the effectiveness of the theoretical results. Finally, Section 6 summarizes the main results.

Notations. $\mathbb{R}$ represents the set of real numbers. $\mathbb{N}_+$ represents the set of positive integer numbers. $\mathbb{R}^{n\times m}$ represents the set of real-valued matrices. $\mathbb{E}$ represents the mathematical expectation operator. $\mathbb{P}$ represents the probability operator. $\|\cdot\|_1$ represents the 1-norm. $\|\cdot\|$ represents the Euclidean norm or 2-norm. A^T represents the transpose of matrix A.

II. Problem formulation

Consider the following heterogeneous MAS:

$$\begin{cases} \dot{x}_i(t) &= A_i x_i(t) + B_i u_i(t), \\ y_i(t) &= C_i x_i(t), \end{cases} \tag{1}$$

where $i = 1, 2, \dots, N$, $x_i(t) \in \mathbb{R}^{n_i}$, $y_i(t) \in \mathbb{R}^p$ are the system state and output of the i-th agent, respectively, $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m_i}$, $C_i \in \mathbb{R}^{p \times n_i}$ are known matrices, $u_i(t) \in \mathbb{R}^{m_i}$ is the control input.

The interaction topology among the agents is described by a directed graph $\mathcal{G}$. $(i, j) \in \mathcal{G}$ means that there is an edge between agent i and agent j, and agent i can receive information from agent j. The adjacency matrix is defined by $\mathcal{H} = [h_{ij}]_{n\times n}$, where $h_{ii} = 0$, $h_{ij} \in \mathbb{N}_+$ if $(j, i) \in \mathcal{G}$, and $h_{ij} = 0$ otherwise. The set of neighbors of agent i is denoted as N_i. The in-degree Λ_i^{in} and out-degree Λ_i^{out} of agent i is denoted as $\Lambda_i^{in} \doteq \sum_{j\in N_i} h_{ij}$ and $\Lambda_i^{out} \doteq \sum_{j\in N_i} h_{ji}$, respectively. $\mathcal{G}$ is balanced if and only if $\Lambda_i^{in} = \Lambda_i^{out}$. In this case, let $\Lambda_i = \Lambda_i^{in} = \Lambda_i^{out}$ for simplicity. The Laplacian matrix is defined as $L = D - \mathcal{H}$, where $D = diag\{\Lambda_i^{in}, i = 1, \dots, N\}$. In this paper, $\mathcal{G}$ is assumed to be balanced and strongly connected.

The work [8] investigates the differentially private output consensus problem of the system (1). To protect the private dataset $\mathcal{P} = \{y_i(0), i = 1, \dots, N\}$ from the external eavesdroppers, [8] introduces the following auxiliary variables $\xi_i \in \mathbb{R}^p$, $i = 1, 2, \dots, N$:

$$\begin{cases} \dot{\xi}_i(t) &= 0,\ t \neq \tau_k, \\ \xi_i(\tau_k) &= \xi_i(\tau_{k-1}) \\ &\quad +\beta_{k-1} \sum_{j\in N_i} h_{ij} \big(\phi_j(\tau_{k-1}) - \xi_i(\tau_{k-1})\big), \end{cases} \tag{2}$$

where the time sequence of information transmission $\{\tau_k\}$ satisfies $0 = \tau_0 < \tau_1 < \cdots < \tau_k < \tau_{k+1} < \cdots$, and $\lim_{k\to\infty} \tau_k = \infty$, $d_k = \tau_k - \tau_{k-1} \in [d_{\min}, d_{\max}]$ with $0 < d_{\min} \le d_{\max}$, $\xi_i(0) = y_i(0)$, $\phi_j(\cdot) = \xi_j(\cdot) + \omega_j(\cdot)$ is the transmitted information from agent j to agent i, each entry of the *private* noise $\omega_j(\cdot) \in \mathbb{R}^p$ satisfies i.i.d and obeys the Laplacian distribution, i.e. $\omega_{j,l}(\cdot) \sim Lap(0, b)$, the distribution function and the associated density function are, respectively, $F(\cdot)$ and $f(x) = \frac{dF(x)}{dx} \neq 0$, $\beta_k > 0$ satisfies $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\sum_{k=0}^{\infty} \beta_k^2 < \infty$. Many examples of β_k can satisfy these two conditions, for example, $\frac{\nu_1}{(k+1)^{\nu_2}}$ with $\nu_1 > 0$ and $\nu_2 \in (0.5, 1]$. For the convenience of analysis, we take $\beta_k = \frac{\nu_1}{k+1}$ and $\nu_1 \in \mathbb{N}_+$.

From (2), it can be observed that each auxiliary variable ξ_i only depends on the initial output $y_i(0)$ of the original system (1), and the noisy information $\phi_j(\tau_k)$ from the neighbours. Thus, the leaking risk of private data can be reduced by transmitting $\xi_j(\tau_k) + \omega_j(\tau_k)$ instead of $y_j(\tau_k) + \omega_j(\tau_k)$.

It should be noted that the transmitted information $\phi_j(\cdot)$ in [8] is infinite-bit. However, in communication field, the network bandwidth is always limited. Therefore, it is necessary to reconsider the differentially private consensus problem for the case of binary-valued communications. In this paper, the transmitted information $\phi_j(\tau_k)$ is processed by the following binary-valued quantizer:

$$s_{ij}(\tau_k) = I_{\{\phi_j(\tau_k)\le\vartheta_{ij}\}}, \quad (3)$$

where $s_{ij}(\tau_k) = (s_{ij,1}(\tau_k),\dots,s_{ij,p}(\tau_k))^T \in \mathbb{R}^p$ is the binary-valued information from agent j to agent i, which is decided by the indicator function $I_{\{\cdot\}}$. That is, $s_{ij,l}(\tau_k)=1$ if $\phi_{j,l}(\tau_k)\le\vartheta_{ij,l}$, and $s_{ij,l}(\tau_k)=0$ otherwise. Constant vectors $\vartheta_{ij}\in\mathbb{R}^p$ are given thresholds, $j\in N_i$. In this context, the observation dataset from the private dataset $\mathcal{P}$ becomes $\mathcal{O}=\{s_{ij}(\tau_k), i=1,\dots,N, j\in N_i, k=0,1,\dots\}$.

Remark 2.1: In real-world applications, thresholds ϑ_{ij} are widely used. For example, the air-fuel ratio (AFR) is a critical parameter determining the performance, efficiency, and emissions of internal combustion engines. If the AFR exceeds the threshold (approximately 14.7:1), the mixture is classified as lean. Otherwise, the mixture is classified as rich [26]. Other examples can be found in the ATM ABR traffic control [27], and identification of binary perceptrons [28], etc.

Based on the binary-valued information $s_{ij}(\tau_k)$, the estimation $\hat{\xi}_{ij}(\tau_k)$ $(j\in N_i)$ of $\xi_j(\tau_k)$ is formulated by the following recursive projection operator:

$$\begin{aligned}\hat{\xi}_{ij}(\tau_k) \;=\; &\Pi_\Omega\Big\{\hat{\xi}_{ij}(\tau_{k-1}) + \tfrac{\nu_1}{k}\big[F\big(\vartheta_{ij}-\hat{\xi}_{ij}(\tau_{k-1})\big)\\ &-s_{ij}(\tau_k)\big]\Big\},\end{aligned} \quad (4)$$

where $\Pi_\Omega(z)$ is the recursive projection operator defined as follows:

$$\Pi_\Omega(z) \doteq \arg\min_{a\in\Omega}\|z-a\|,\ \forall z\in\mathbb{R}^p,$$

where $\Omega \doteq \{a\in\mathbb{R}^p : \|a\|\le M\}$.

By virtue of (4), the second equation of (2) can be reconstructed as follows:

$$\xi_i(\tau_k) = \xi_i(\tau_{k-1}) + \frac{\nu_1}{k}\sum_{j\in N_i} h_{ij}\big(\hat{\xi}_{ij}(\tau_{k-1})-\xi_i(\tau_{k-1})\big). \quad (5)$$

To achieve the output consensus target, the work [8] adopts the continuous control strategy, which means that the control signal is constantly generated, and feeds back into the original system continuously. In this context, the communication resources may be wasted. An interesting question is whether the consensus target can also be achieved via the impulsive control protocol, by which the control signals are generated and affect the original system only at some impulsive instants.

To answer the above question, we design the following impulsive controller:

$$u_i(t) = \sum_{k=0}^{\infty}\Big[K_{1,i}x_i(t^-)+K_{2,i}\xi_i(t^-)\Big]\delta(t-\tau_k), \quad (6)$$

where $k\in\mathbb{N}$, $K_{1,i}$ and $K_{2,i}$ are the impulsive control gains, $x_i(t^-)$ and $\xi_i(t^-)$ denote the left limits of x_i and ξ_i at t, respectively, $\delta(\cdot)$ is the Dirac delta function, and τ_k is from (2).

Then, by (6) we have

$$\begin{cases} u_i(t) &= 0,\ t\neq\tau_k,\\ u_i(\tau_k) &= K_{1,i}x_i(\tau_k^-)+K_{2,i}\xi_i(\tau_k^-).\end{cases} \quad (7)$$

Moreover, by (7), (1) can be transformed into

$$\begin{cases} \dot{x}_i(t) &= A_ix_i(t),\ t\neq\tau_k,\\ x_i(\tau_k^+) &= x_i(\tau_k^-)+B_iu_i(\tau_k)\\ &= (I+B_iK_{1,i})x_i(\tau_k^-)+B_iK_{2,i}\xi_i(\tau_k^-),\end{cases} \quad (8)$$

where $x_i(\tau_k^+)$ and $x_i(\tau_k^-)$ denote the right and left limits of x_i at τ_k, respectively. In this paper, it is assumed that for all $k\in\mathbb{N}$, $x_i(\tau_k^+)=x_i(\tau_k)$ and $\xi_i(\tau_k^+)=\xi_i(\tau_k)$.

By (8) we have

$$x_i(\tau_k) = (I+B_iK_{1,i})e^{A_id_k}x_i(\tau_{k-1})+B_iK_{2,i}\xi_i(\tau_{k-1}), \quad (9)$$

The following assumptions are important for the main results.

Assumption 2.2: (see [8]) For any $i=1,\dots,N$, there exists a solution (Θ_i,U_i) of the following equation:

$$\begin{cases} A_i\Theta_i+B_iU_i &= 0,\\ C_i\Theta_i &= I.\end{cases}$$

Assumption 2.3: (i) System (8) is impulsively controllable [29]. (ii) There exists a solution $(K_{1,i},K_{2,i})$ such that $\big((I+B_iK_{1,i})e^{A_id_k}-I\big)\Theta_i+B_iK_{2,i}=0$ for any $i=1,\dots,N$.

Remark 2.4: The condition (ii) in Assumption 2.3 is necessary for the main results in Section III. If this condition is not satisfied, the consensus error will not asymptotically converge to zero in the mean square sense. If B_i is of full row rank for any $i=1,\dots,N$, then we can design $K_{2,i}=B_i^T(B_iB_i^T)^{-1}\big(I-(I+B_iK_{1,i})e^{A_id_k}\big)\Theta_i$ to ensure that (ii) of Assumption 2.3 holds. When B_i is not of full row rank, the matrix $K_{2,i}$ may not be explicitly constructed. But we can still get $K_{2,i}$ by solving the condition (ii) in Assumption 2.3. For more details, please refer to Section V.

This paper aims to achieve the output consensus of the system (1) while protecting the private dataset $\mathcal{P}$.

Definition 2.5: System (1) is said to achieve the mean square output consensus under the controller (6), if for any i, it holds $\lim_{k\to\infty}\mathbb{E}\big[\|y_i(\tau_k)-\frac{1}{N}\sum_{i=1}^N\xi_i(\tau_k)\|^2\big]=0$.

Remark 2.6: By noting $\xi_i(0)=y_i(0)$ and from (5), we can get that $\xi_i(\tau_k)$ depends on $y_i(0)$. Therefore, it is important to protect the private dataset $\mathcal{P}$ since the consensus target is related to $\mathcal{P}$.

For any $k\in\mathbb{N}$, by (1) and Assumption 2.2, we have

$$\begin{aligned} &y_i(\tau_k)-\tfrac{1}{N}\textstyle\sum_{i=1}^N\xi_i(\tau_k)\\ =\;& C_ix_i(\tau_k)-\tfrac{1}{N}\textstyle\sum_{i=1}^N\xi_i(\tau_k)\\ =\;& C_ix_i(\tau_k)-\xi_i(\tau_k)+\xi_i(\tau_k)-\tfrac{1}{N}\textstyle\sum_{i=1}^N\xi_i(\tau_k)\\ =\;& C_i(x_i(\tau_k)-\Theta_i\xi_i(\tau_k))+\xi_i(\tau_k)-\tfrac{1}{N}\textstyle\sum_{i=1}^N\xi_i(\tau_k).\end{aligned} \quad (10)$$

Let $x_{ci}(\tau_k)\doteq x_i(\tau_k)-\Theta_i\xi_i(\tau_k)$ and $\tilde{\xi}_i(\tau_k)\doteq\xi_i(\tau_k)-\frac{1}{N}\sum_{i=1}^N\xi_i(\tau_k)$. Then, if $\lim_{k\to\infty}\mathbb{E}\big[\|x_{ci}(\tau_k)\|^2\big]=0$ and $\lim_{k\to\infty}\mathbb{E}\big[\|\tilde{\xi}_i(\tau_k)\|^2\big]=0$, then it holds $\lim_{k\to\infty}\mathbb{E}\big[\|y_i(\tau_k)-\frac{1}{N}\sum_{i=1}^N\xi_i(\tau_k)\|^2\big]=0$. In the forthcoming sections, we mainly analyze $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$.

III. Convergence and convergence rate

This section investigates the convergence and convergence rates of $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$. To this end, some useful transformations of $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$ are needed as follows.

According to (5), (9) and the definition of $x_{ci}(\tau_k)$ below (10), we can obtain

$$\begin{aligned} & x_{ci}(\tau_k) \\ = \ & x_i(\tau_k) - \Theta_i \xi_i(\tau_k) \\ = \ & (I + B_i K_{1,i}) e^{A_i d_k} x_i(\tau_{k-1}) + (B_i K_{2,i} - \Theta_i)\xi_i(\tau_{k-1}) \\ & -\tfrac{\nu_1 \Theta_i}{k} \textstyle\sum_{j\in N_i} h_{ij}\big(\hat{\xi}_{ij}(\tau_{k-1}) - \xi_i(\tau_{k-1})\big) \\ = \ & (I + B_i K_{1,i}) e^{A_i d_k} x_{ci}(\tau_{k-1}) \\ & + \big[(I + B_i K_{1,i}) e^{A_i d_k}\Theta_i + B_i K_{2,i} - \Theta_i\big]\xi_i(\tau_{k-1}) \\ & -\tfrac{\nu_1 \Theta_i}{k} \textstyle\sum_{j\in N_i} h_{ij}\big(\hat{\xi}_{ij}(\tau_{k-1}) - \xi_i(\tau_{k-1})\big). \end{aligned} \quad (11)$$

From Assumption 2.3 (ii), we can transform (11) into

$$\begin{aligned} x_{ci}(\tau_k) = \ & (I + B_i K_{1,i}) e^{A_i d_k} x_{ci}(\tau_{k-1}) \\ & -\tfrac{\nu_1 \Theta_i}{k} \textstyle\sum_{j\in N_i} h_{ij}\big(\hat{\xi}_{ij}(\tau_{k-1}) - \xi_i(\tau_{k-1})\big). \end{aligned} \quad (12)$$

Let $x_c = (x_{c1}, \dots, x_{cN})^T$, $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_N)^T$, $A = diag\{A_1, \dots, A_N\}$, $B = diag\{B_1, \dots, B_N\}$, $K_1 = diag\{K_{1,1}, \dots, K_{1,N}\}$, $\Theta = diag\{\Theta_1, \dots, \Theta_N\}$, $\varepsilon = (\varepsilon_{1r_1}, \dots, \varepsilon_{1r_{\Lambda_1}}, \varepsilon_{2r_{\Lambda_1+1}}, \dots, \varepsilon_{2r_{\Lambda_1+\Lambda_2}}, \dots, \varepsilon_{Nr_{\Lambda_1+\dots+\Lambda_{N-1}+1}}, \dots, \varepsilon_{Nr_{\Lambda_1+\dots+\Lambda_N}})^T$, $\varepsilon_{ij} = \hat{\xi}_{ij} - \xi_j$. Then, by (12) and the definition of $\tilde{\xi}_i(\tau_k)$ below (10), we have

$$\begin{cases} x_c(\tau_{k+1}) = (I + BK_1) e^{A d_{k+1}} x_c(\tau_k) \\ \qquad + \frac{\nu_1 \Theta}{k+1}\Big[(L\otimes I_p)\tilde{\xi}(\tau_k) - (\tilde{H}\otimes I_p)\varepsilon(\tau_k)\Big], \\ \tilde{\xi}(\tau_{k+1}) = \big[(I_N - \frac{\nu_1}{k+1}L)\otimes I_p\big]\tilde{\xi}(\tau_k) \\ \qquad + \frac{\nu_1}{k+1}(J\tilde{H}\otimes I_p)\varepsilon(\tau_k), \end{cases} \quad (13)$$

where L is the Laplacian matrix, $\tilde{H} = (\tilde{h}_{1r_1}, \dots, \tilde{h}_{1r_{\Lambda_1}}, \dots, \tilde{h}_{Nr_{\Lambda_1+\dots+\Lambda_{N-1}+1}}, \dots, \tilde{h}_{Nr_{\Lambda_1+\dots+\Lambda_N}})$ with $\tilde{h}_{ij} = (0, \dots, \underbrace{h_{ij}}_{i\ th\ position}, \dots, 0)^T$, h_{ij} is the element of the adjacency matrix $\mathcal{H}$, and $J = I_N - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ with $\mathbf{1} = (1, \dots, 1)^T$.

Before proceeding with the convergence and convergence rates of $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$, we need the following boundedness result for $\xi_i(\tau_k)$.

Lemma 3.1: For any $k \geq \nu_1 \Lambda_i$ and any $i = 1, \dots, N$, the inequality $\|\xi_i(\tau_k)\| \leq M$ holds.

Proof. From (5), we have

$$\xi_i(\tau_k) = (1 - \frac{\nu_1 \Lambda_i}{k})\xi_i(\tau_{k-1}) + \frac{\nu_1}{k}\sum_{j\in N_i} h_{ij}\hat{\xi}_{ij}(\tau_{k-1}). \quad (14)$$

By (4), we can get that $\|\hat{\xi}_{ij}(\tau_k)\| \leq M$ for any k. Thus, when $k = \nu_1 \Lambda_i$, (14) yields $\|\xi_i(\tau_k)\| \leq M$.

Let $\|\xi_i(\tau_{k^*})\| \leq M$ hold for some $k^* > \nu_1 \Lambda_i$. Then, by (14), we can get that

$$\begin{aligned} \|\xi_i(\tau_{k^*+1})\| & \leq |1 - \tfrac{\nu_1 \Lambda_i}{k^*+1}|\|\xi_i(\tau_{k^*})\| + \tfrac{\nu_1}{k^*+1}\Lambda_i M \\ & \leq (1 - \tfrac{\nu_1 \Lambda_i}{k^*+1})M + \tfrac{\nu_1}{k^*+1}\Lambda_i M \\ & = M. \end{aligned}$$

By the mathematical induction method, the inequality $\|\xi_i(\tau_k)\| \leq M$ holds for any $k \geq \nu_1 \Lambda_i$ and any $i = 1, \dots, N$. The proof is completed. ∎

Remark 3.2: For the convenience of analysis, it is required that $h_{ij} \in \mathbb{N}_+$ for $(j, i) \in \mathcal{G}$, and $\nu_1 \in \mathbb{N}_+$. For the case that $h_{ij} > 0$ and $\nu_1 > 0$, Lemma 3.1 can also be proved by taking proper ν_1 such that $\nu_1 \Lambda_i \in \mathbb{N}_+$ for any i.

Construct three Lyapunov functions as $V_1(k) = \mathbb{E}\big[\tilde{\xi}^T(\tau_k)\tilde{\xi}(\tau_k)\big]$, $V_2(k) = \mathbb{E}\big[x_c^T(\tau_k)x_c(\tau_k)\big]$, $V_3(k) = \mathbb{E}\big[\varepsilon^T(\tau_k)\varepsilon(\tau_k)\big]$. Based on Lemma 3.1, we can obtain the following three lemmas concerning the relationship among $V_i(k)$ $(i = 1, 2, 3)$, which are important to derive the convergence and convergence rates of $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$.

Lemma 3.3: For any $k \geq \nu_1 \Lambda_{\max}$, where $\Lambda_{\max} = \max\{\Lambda_i, i = 1, \dots, N\}$, the following inequality holds:

$$\begin{aligned} & V_1(k+1) \\ \leq \ & \Big[1 - \tfrac{2\nu_1}{k+1}\lambda_2(\bar{L}) + \tfrac{\nu_1^2\|L\|^2}{(k+1)^2} + \tfrac{\nu_1\gamma_1}{k+1}\|\Psi_2\|^2\Big]V_1(k) \\ & + \tfrac{\nu_1}{\gamma_1(k+1)}V_3(k) + \tfrac{1}{(k+1)^2}\Delta_1, \end{aligned} \quad (15)$$

where $\bar{L} = \frac{L^T + L}{2}$ with $\lambda_2(\bar{L}) \leq \lambda_3(\bar{L}) \leq \dots \leq \lambda_N(\bar{L})$, $\Psi_2 = (J - \frac{\nu_1}{k+1}L)\tilde{H}\otimes I_p$, $\gamma_1 > 0$ is an arbitrary constant and $\Delta_1 > 0$ is a constant.

Proof. From (13), and by noting $JL = LJ = L$, we have

$$\begin{aligned} V_1(k+1) = \ & \mathbb{E}\Big\{\tilde{\xi}^T(\tau_k)\big[(I_N - \tfrac{\nu_1}{k+1}L)^2\otimes I_p\big]\tilde{\xi}(\tau_k) \\ & + \tfrac{2\nu_1}{k+1}\tilde{\xi}^T(\tau_k)\big[(J - \tfrac{\nu_1}{k+1}L)\tilde{H}\otimes I_p\big]\varepsilon(\tau_k) \\ & + \tfrac{\nu_1^2}{(k+1)^2}\varepsilon^T(\tau_k)(\tilde{H}^T J^T J\tilde{H}\otimes I_p)\varepsilon(\tau_k)\Big\}. \end{aligned} \quad (16)$$

By Lemma 3.1, and by noting $\varepsilon_{ij} = \hat{\xi}_{ij} - \xi_j$, we know that $\varepsilon(\tau_k)$ is bounded for any $k \geq \nu_1 \Lambda_{\max}$. Then, (16) yields

$$\begin{aligned} & V_1(k+1) \\ \leq \ & \mathbb{E}\Big\{\tilde{\xi}^T(\tau_k)\big[(I_N - \tfrac{\nu_1}{k+1}L)^2\otimes I_p\big]\tilde{\xi}(\tau_k) \\ & + \tfrac{2\nu_1}{k+1}\tilde{\xi}^T(\tau_k)\big[(J - \tfrac{\nu_1}{k+1}L)\tilde{H}\otimes I_p\big]\varepsilon(\tau_k) + \tfrac{\Delta_1}{(k+1)^2}\Big\} \\ \leq \ & \Big[1 - \tfrac{2\nu_1}{k+1}\lambda_2(\bar{L}) + \tfrac{\nu_1^2\|L\|^2}{(k+1)^2} + \tfrac{\nu_1\gamma_1}{k+1}\|\Psi_2\|^2\Big]V_1(k) \\ & + \tfrac{\nu_1}{\gamma_1(k+1)}V_3(k) + \tfrac{1}{(k+1)^2}\Delta_1. \end{aligned}$$

The proof is completed. ∎

Lemma 3.4: For any $k \geq \nu_1 \Lambda_{\max}$, the following inequality holds:

$$\begin{aligned} V_2(k+1) \leq \ & \tfrac{\nu_1}{\gamma_2(k+1)}\|\Theta(L\otimes I_p)\|^2 V_1(k) \\ & + \big(1 + \tfrac{\nu_1(\gamma_2+\gamma_3)}{k+1}\big)\|\Phi\|^2 V_2(k) \\ & + \tfrac{\nu_1}{\gamma_3(k+1)}\|\Theta(\mathcal{H}\otimes I_p)\|^2 V_3(k) + \tfrac{1}{(k+1)^2}\Delta_2, \end{aligned} \quad (17)$$

where $\Phi=(I+BK_1)e^{Ad_{k+1}}$, $\gamma_2>0$ and $\gamma_3>0$ are arbitrary constants, and $\Delta_2>0$ is a constant.

Proof. Similar to the proof of Lemma 3.3, we can get

$$\begin{aligned}V_2(k+1)\ \le\ & \mathbb{E}\Big\{x_c^T(\tau_k)\Phi^T\Phi x_c(\tau_k)\\ &+\tfrac{2\nu_1}{k+1}x_c^T(\tau_k)\Phi^T\Theta(L\otimes I_p)\tilde{\xi}(\tau_k)\\ &-\tfrac{2\nu_1}{k+1}x_c^T(\tau_k)\Phi^T\Theta(\mathcal{H}\otimes I_p)\varepsilon(\tau_k)+\tfrac{\Delta_2}{(k+1)^2}\Big\}\\ \le\ & \tfrac{\nu_1}{\gamma_2(k+1)}\|\Theta(L\otimes I_p)\|^2V_1(k)\\ &+\big(1+\tfrac{\nu_1(\gamma_2+\gamma_3)}{k+1}\big)\|\Phi\|^2V_2(k)\\ &+\tfrac{\nu_1}{\gamma_3(k+1)}\|\Theta(\mathcal{H}\otimes I_p)\|^2V_3(k)+\tfrac{\Delta_2}{(k+1)^2}.\end{aligned}$$

The proof is completed. ∎

Lemma 3.5: For any $k\ge\nu_1\Lambda_{\max}$, the following inequality holds:

$$\begin{aligned}&V_3(k+1)\\ \le\ & \tfrac{\nu_1\|\Psi_4\|^2}{\gamma_4(k+1)}V_1(k)+\big[(1+\tfrac{\nu_1\gamma_4}{k+1})(1-\tfrac{\nu_1\lambda_{\min}(\Psi_3^T+\Psi_3)}{k+1}\\ &+\tfrac{\nu_1^2\|\Psi_3\|^2}{(k+1)^2})-\tfrac{2\nu_1 f_m}{k+1}\big]V_3(k)+\tfrac{\Delta_3}{(k+1)^2},\end{aligned}\tag{18}$$

where $\Psi_3=(Q\otimes I_p)(\tilde{H}\otimes I_p)$, $\Psi_4=(Q\otimes I_p)(L\otimes I_p)$, $f_m=\min_{l=1,\ldots,p}\{f_l(\vartheta_{\max}+M)\}$ with $\vartheta_{\max}=\max_{i=1,\ldots,N,j\in N_i}\{\vartheta_{ij}\}$, $\gamma_4>0$ is an arbitrary constant, and $\Delta_3>0$ is a constant.

Proof. By Lemma 3.1, $\|\xi_j(\tau_k)\|\le M$ holds for any j and any $k\ge\nu_1\Lambda_{\max}$. Thus, $x_j(\tau_k)=\Pi_M(x_j(\tau_k))$ holds for any j and any $k\ge\nu_1\Lambda_{\max}$. From (4) to (5), we have

$$\begin{aligned}&\|\varepsilon_{ij}(\tau_{k+1})\|\\ =\ & \|\hat{\xi}_{ij}(\tau_{k+1})-\xi_j(\tau_{k+1})\|\\ =\ & \Big\|\Pi_\Omega\big\{\hat{\xi}_{ij}(\tau_k)+\tfrac{\nu_1}{k+1}\big(F\big(\vartheta_{ij}-\hat{\xi}_{ij}(\tau_k)\big)\\ &-s_{ij}(\tau_{k+1})\big)\big\}-\Pi_\Omega\big(\xi_j(\tau_{k+1})\big)\Big\|\\ \le\ & \Big\|\varepsilon_{ij}(\tau_k)+\tfrac{\nu_1}{k+1}\big(F\big(\vartheta_{ij}-\hat{\xi}_{ij}(\tau_k)\big)-s_{ij}(\tau_{k+1})\big)\\ &-\tfrac{\nu_1}{k+1}\sum_{l\in N_j}h_{jl}(\hat{\xi}_{jl}(\tau_k)-\xi_j(\tau_k))\Big\|,\ k\ge\nu_1\Lambda_{\max},\end{aligned}$$

which yields

$$\begin{aligned}&V_3(k+1)\\ \le\ & \mathbb{E}\Big\{\varepsilon^T(\tau_k)\big\|I-\tfrac{\nu_1}{k+1}(Q\otimes I_p)(\tilde{H}\otimes I_p)\big\|^2\varepsilon(\tau_k)\\ &+\tfrac{\nu_1^2}{(k+1)^2}\tilde{\xi}^T(\tau_k)\big\|(Q\otimes I_p)(L\otimes I_p)\big\|^2\tilde{\xi}(\tau_k)\\ &+\tfrac{\nu_1^2}{(k+1)^2}\big\|F\big(\vartheta-\hat{\xi}(\tau_k)\big)-s(\tau_{k+1})\big\|^2\\ &+\tfrac{2\nu_1}{k+1}\varepsilon^T(\tau_k)\big[I-\tfrac{\nu_1}{k+1}(Q\otimes I_p)(\tilde{H}\otimes I_p)\big]^T\\ &\times(Q\otimes I_p)(L\otimes I_p)\tilde{\xi}(\tau_k)\\ &+\tfrac{2\nu_1}{k+1}\varepsilon^T(\tau_k)\big[I-\tfrac{\nu_1}{k+1}(Q\otimes I_p)(\tilde{H}\otimes I_p)\big]^T\\ &\times\big[F\big(\vartheta-\hat{\xi}(\tau_k)\big)-s(\tau_{k+1})\big]\\ &+\tfrac{2\nu_1^2}{(k+1)^2}\tilde{\xi}^T(\tau_k)(L^T\otimes I_p)(Q^T\otimes I_p)\\ &\times\big[F\big(\vartheta-\hat{\xi}(\tau_k)\big)-s(\tau_{k+1})\big]\Big\},\ k\ge\nu_1\Lambda_{\max},\end{aligned}\tag{19}$$

where $Q=\big(q_{1r_1},\ldots,q_{1r_{\Lambda_1}},\ldots,q_{Nr_{\Lambda_1+\ldots+\Lambda_{N-1}+1}},\ldots,q_{Nr_{\Lambda_1+\ldots+\Lambda_N}}\big)^T$ with $q_{ij}=(0,\ldots,\underbrace{1}_{j\ th\ position},\ldots,0)^T$, and $\vartheta=\big(\vartheta_{1r_1},\ldots,\vartheta_{1r_{\Lambda_1}},\ldots,\vartheta_{Nr_{\Lambda_1+\ldots+\Lambda_{N-1}+1}},\ldots,\vartheta_{Nr_{\Lambda_1+\ldots+\Lambda_N}}\big)^T$.

Next, we mainly analyze the term $\frac{2\nu_1}{k+1}\varepsilon^T(\tau_k)\big[I-\frac{\nu_1}{k+1}(Q\otimes I_p)(\tilde{H}\otimes I_p)\big]^T\big[F\big(\vartheta-\hat{\xi}(\tau_k)\big)-s(\tau_{k+1})\big]$. The other terms in (19) can be either related to $V_i(k)$ $(i=1,2,3)$ or rewritten as $\frac{\Delta_4}{(k+1)^2}$ with $\Delta_4>0$ being a constant.

Define a σ-algebra as $\mathcal{D}_k=\sigma\{y_i(0),\omega_i(\tau_l),i=1,\ldots,N,l=1,\ldots,k\}$. Then both $\hat{\xi}_{ij}(\tau_k)$ and $\xi_j(\tau_k)$ are $\mathcal{D}_k$ measurable, which yield $\varepsilon(\tau_k)$ is also $\mathcal{D}_k$ measurable.

Similar to the proofs of Lemma 3.5 in [19] and Lemma 2 in [30], we have

$$\begin{aligned}&\mathbb{E}\Big\{\tfrac{2\nu_1}{k+1}\varepsilon^T(\tau_k)\big[I-\tfrac{\nu_1}{k+1}(Q\otimes I_p)(\tilde{H}\otimes I_p)\big]^T\\ &\times\big[F\big(\vartheta-\hat{\xi}(\tau_k)\big)-s(\tau_{k+1})\big]\Big\}\\ \le\ & -\tfrac{2\nu_1}{k+1}f_mV_3(k)+\tfrac{\Delta_5}{(k+1)^2},\end{aligned}\tag{20}$$

where $\Delta_5>0$ is a constant.

Thus, by (19) and (20), we can obtain

$$\begin{aligned}&V_3(k+1)\\ \le\ & \tfrac{\nu_1\|\Psi_4\|^2}{\gamma_4(k+1)}V_1(k)+\Big[(1+\tfrac{\nu_1\gamma_4}{k+1})(1-\tfrac{\nu_1\lambda_{\min}(\Psi_3^T+\Psi_3)}{k+1}\\ &+\tfrac{\nu_1^2\|\Psi_3\|^2}{(k+1)^2})-\tfrac{2\nu_1 f_m}{k+1}\Big]V_3(k)+\tfrac{\Delta_3}{(k+1)^2},\end{aligned}$$

The proof is completed. ∎

Now, based on Lemmas 3.3-3.5, we are in a position to propose the following theorem concerning the convergence and convergence rates of $x_{ci}(\tau_k)$ and $\tilde{\xi}_i(\tau_k)$.

Theorem 3.6: Let $V(k)=\sum_{i=1}^3V_i(k)$. Then, the following result holds as $k\to\infty$:

$$V(k)=\begin{cases}O\big(\frac{1}{k^{\nu_1\Psi_5}}\big), & if\ \ 0<\nu_1\Psi_5<1,\\ O\big(\frac{\ln k}{k}\big), & if\ \ \nu_1\Psi_5=1,\\ O\big(\frac{1}{k}\big), & if\ \ \nu_1\Psi_5>1,\end{cases}$$

where $\Psi_5=2\lambda_2(\bar{L})-\frac{\|\Theta(L\otimes I_p)\|^2}{\gamma_2}-\frac{\|\Psi_4\|^2}{\gamma_4}-\gamma_1\|\Psi_2\|^2$, γ_1, Ψ_2 and $\bar{L}$ are from Lemma 3.3, γ_2 is from Lemma 3.4, γ_4 and Ψ_4 are from Lemma 3.5, Θ is below (12).

Proof. From Lemmas 3.3-3.5, we have

$$\begin{aligned}&V(k+1)\\ \le\ & \Big[1-\tfrac{2\nu_1}{k+1}\lambda_2(\bar{L})+\tfrac{\nu_1^2\|L\|^2}{(k+1)^2}+\tfrac{\nu_1\gamma_1}{k+1}\|\Psi_2\|^2\end{aligned}\tag{21}$$

$$
\begin{aligned}
&+\tfrac{\nu_1}{\gamma_2(k+1)}\|\Theta(L\otimes I_p)\|^2+\tfrac{\nu_1}{\gamma_4(k+1)}\|\Psi_4\|^2\Big]V_1(k)\\
&+\big(1+\tfrac{\nu_1(\gamma_2+\gamma_3)}{k+1}\big)\|\Phi\|^2V_2(k)\\
&+\Big[(1+\tfrac{\nu_1\gamma_4}{k+1})(1-\tfrac{\nu_1\lambda_{\min}(\Psi_3^T+\Psi_3)}{k+1}+\tfrac{\nu_1^2}{(k+1)^2}\|\Psi_3\|^2)\\
&-\tfrac{2\nu_1 f_m}{k+1}+\tfrac{\nu_1}{\gamma_1(k+1)}+\tfrac{\nu_1}{\gamma_3(k+1)}\|\Theta(\mathcal{H}\otimes I_p)\|^2\Big]V_3(k)\\
&+\tfrac{\Delta_4}{(k+1)^2}\\
\doteq\ & \alpha_1(k)V_1(k)+\alpha_2(k)V_2(k)+\alpha_3(k)V_3(k)+\tfrac{\Delta_4}{(k+1)^2},
\end{aligned}
$$

where $\Delta_4 > 0$ is a constant.

From Assumption 2.3, and by noting $d_k \in [d_{\min}, d_{\max}]$, we can select appropriate parameters γ_i $(i = 1, 2, 3, 4)$ and K_1 such that $\max\{\alpha_2(k), \alpha_3(k)\} \le \alpha_1(k)$. Thus, (21) yields

$$
V(k+1) \le \Big[1-\frac{\nu_1\Psi_5}{k+1}+\frac{\nu_1^2\|L\|^2}{(k+1)^2}\Big]V(k)+\frac{\Delta_4}{(k+1)^2}. \quad (22)
$$

Since $\hat{\xi}_{ij}(\tau_k)$ and $\xi_j(\tau_k)$ are bounded, $V_i(k)$ $(i = 1, 2, 3)$ and $V(k)$ are also bounded. Therefore, (22) yields

$$
V(k+1) \le \Big(1-\frac{\nu_1\Psi_5}{k+1}\Big)V(k)+\frac{\Delta_6}{(k+1)^2}, \quad (23)
$$

where $\Delta_6 > 0$ is a constant.

By Lemma 3.8 in [19] and Lemma 4 in [14], we have

$$
V(k)=\begin{cases} O\big(\frac{1}{k^{\nu_1\Psi_5}}\big), & 0<\nu_1\Psi_5<1,\\ O\big(\frac{\ln k}{k}\big), & \nu_1\Psi_5=1,\\ O\big(\frac{1}{k}\big), & \nu_1\Psi_5>1. \end{cases}
$$

The proof is completed. ∎

Remark 3.7: By Theorem 3.6, we can get that $\lim_{k\to\infty}\mathbb{E}\big[\|x_{ci}(\tau_k)\|^2\big]=0$ and $\lim_{k\to\infty}\mathbb{E}\big[\|\tilde{\xi}_i(\tau_k)\|^2\big]=0$, thereby yielding $\lim_{k\to\infty}\mathbb{E}\big[\|y_i(\tau_k)-\frac{1}{N}\sum_{i=1}^N\xi_i(\tau_k)\|^2\big]=0$. If the impulsive controller (6) becomes the state-feedback form $u_i(t)=K_{1,i}x_i(t)+K_{2,i}\xi_i(t)$ as shown in [8], then it holds $\lim_{t\to\infty}\mathbb{E}\big[\|y_i(t)-\frac{1}{N}\sum_{i=1}^N\xi_i(t)\|^2\big]=0$.

Remark 3.8: The main novelty of this paper lies in three aspects: privacy noise design, communication scheme, and control strategy. (i) The privacy noise ω in (2) is non-decaying, providing stronger privacy protection compared to the exponentially decaying noises used in [16], [17]. (ii) The 1-bit communication scheme (3) significantly reduces resource consumption and bandwidth requirements compared to the infinite-bit transmission scheme in [8]. (iii) The generation, processing and transmission of control signals always consume power. In the designed impulsive control protocol (6), control signals are generated and affect the original system only at some impulsive instants. Thus, the control protocol (6) can save communication resources compared to the continuous state-feedback strategy in [8].

IV. Privacy analysis

In this section, we investigate the differential privacy index of the proposed algorithm. In Section 2, we define a private dataset as $\mathcal{P} = \{y_i(0), i = 1, \ldots, N\}$. Before giving the definition of differential privacy, we define another private dataset as $\mathcal{P}' = \{y'_i(0), i = 1, \ldots, N\}$.

Definition 4.1: ([6], [8]) Given a time horizon $T > 0$ and a parameter $\varepsilon > 0$. For any subset $\mathcal{O}_1 \subseteq \mathbb{R}^n$ and any two datasets $\mathcal{D}$ and $\mathcal{D}'$, a randomized mechanism $\mathcal{M} : \mathcal{D}(\mathcal{D}') \to \mathcal{O}_1$ is said to be ε-differentially private up to time $T-1$, if it holds that $\mathbb{P}\big[\mathcal{M}(\mathcal{D}) \in \mathcal{O}_1\big] \le e^{\varepsilon\|\mathcal{D}-\mathcal{D}'\|_1}\mathbb{P}\big[\mathcal{M}(\mathcal{D}') \in \mathcal{O}_1\big]$.

To obtain the differential privacy index, we need the sensitivity of a randomized mechanism as follows.

Definition 4.2: ([8]) The sensitivity of a randomized mechanism is

$$
S(\tau_k)=\sup_{\mathcal{P},\mathcal{P}',\mathcal{O}}\frac{\|\rho(\mathcal{P},\mathcal{O})(\tau_{k-1})-\rho(\mathcal{P}',\mathcal{O})(\tau_{k-1})\|_1}{\|\mathcal{P}-\mathcal{P}'\|_1}, \quad (24)
$$

where $\rho(\mathcal{P},\mathcal{O})(\tau_k) = \{\xi_i^{\mathcal{P},\mathcal{O}}(\tau_k), i = 1, 2, \ldots, N\}$ and $\rho(\mathcal{P}',\mathcal{O})(\tau_k) = \{\xi_i^{\mathcal{P}',\mathcal{O}}(\tau_k), i = 1, 2, \ldots, N\}$, $\xi_i^{\mathcal{P},\mathcal{O}}(\tau_k)$ is from (5) and is related to the private dataset $\mathcal{P}$ and the observation dataset $\mathcal{O}$, which is given below (3). Correspondingly, $\xi_i^{\mathcal{P}',\mathcal{O}}(\tau_k)$ is related to the private dataset $\mathcal{P}'$ and the observation dataset $\mathcal{O}$.

Theorem 4.3: The sensitivity $S(\tau_k)$ satisfies

$$
S(\tau_k)=\begin{cases} 1, & k=1,\\ \prod_{l=0}^{k-2}\big(\frac{\nu_1\Lambda_{\max}}{l+1}-1\big), & 2\le k\le \nu_1\Lambda_{\max},\\ 0, & k>\nu_1\Lambda_{\max}, \end{cases} \quad (25)
$$

where ν_1 is from (2), $\Lambda_{\max} = \max\{\Lambda_i, i = 1, \ldots, N\}$.

Proof. For the private datasets $\mathcal{P}$ and $\mathcal{P}'$, the generated observation datasets $\mathcal{O}$ are the same, i.e. $\mathcal{O} = \{s_{ij}(\tau_k), i = 1, \ldots, N, j \in N_i, k = 0, 1, \ldots\}$. Thus, the estimations $\hat{\xi}_{ij}(\tau_k)$ $(i = 1, 2, \ldots, N, j \in N_i)$ can be constructed as (4) for both $\mathcal{P}$ and $\mathcal{P}'$. Then, by (5) we have

$$
\begin{aligned}
&\xi_i^{\mathcal{P},\mathcal{O}}(\tau_{k-1})-\xi_i^{\mathcal{P}',\mathcal{O}}(\tau_{k-1})\\
=\ &\big(1-\tfrac{\nu_1\Lambda_i}{k-1}\big)\big(\xi_i^{\mathcal{P},\mathcal{O}}(\tau_{k-2})-\xi_i^{\mathcal{P}',\mathcal{O}}(\tau_{k-2})\big).
\end{aligned} \quad (26)
$$

For each i and any $k > \nu_1\Lambda_i$, by (26) we have $\xi_i^{\mathcal{P},\mathcal{O}}(\tau_{k-1}) - \xi_i^{\mathcal{P}',\mathcal{O}}(\tau_{k-1}) \equiv 0$. Thus, for $k > \nu_1\Lambda_{\max}$, it holds $S(\tau_k) = 0$. The rest of proof is similar to that of Theorem 3.4 in [8], so it is omitted here. The proof is completed. ∎

The following theorem shows the differential privacy index ε based on Theorem 4.3.

Theorem 4.4: The differential privacy index ε over time horizon T satisfies $\varepsilon = \frac{\sum_{k=1}^T S(\tau_k)}{b}$, where b is from (2).

Proof. The proof is similar to that of Theorem 3.5 in [8], so it is omitted here. ∎

Remark 4.5: In [8], $S(\tau_k)$ is defined as follows

$$
S(\tau_k)=\begin{cases} 1, & k=1,\\ \prod_{l=0}^{k-2}(1-\beta_l\Lambda_{\min}), & k\ge 2, \end{cases} \quad (27)
$$

where $\Lambda_{\min} = \min\{\Lambda_i, i = 1, \ldots, N\}$, and β_k satisfies (i) $\sum_{k=0}^{\infty}\beta_k = \infty$, (ii) $\sum_{k=0}^{\infty}\beta_k^2 < \infty$, (iii) $\beta_k < \frac{1}{\Lambda_{\max}}$. Based on (27), the differential privacy index ε is also derived as $\varepsilon = \frac{\sum_{k=1}^T S(\tau_k)}{b}$. In Theorem 4.3, we only require β_k satisfies (i) and (ii). If β_k also satisfies (iii), then (25) will become (27). In some contexts, ε in this paper is smaller than that in [8]. For example, let $\Lambda_i \equiv 1$ for any i and T be sufficiently large. In (25), let $\nu_1 = 10$. Then, $S(\tau_k) = \prod_{l=0}^{k-2}(\frac{10}{l+1} - 1)$ for $2 \leq k \leq 10$ and $S(\tau_k) \equiv 0$ for $k > 10$, which yield $\varepsilon = \frac{1}{b}\big[1+\sum_{k=2}^{10}\prod_{l=0}^{k-2}(\frac{10}{l+1}-1)\big] < \infty$. In (27), let $\beta_k = \frac{1}{k+2}$. Then, $S(\tau_k) = \frac{1}{k}$ and $\varepsilon = \frac{1}{b}\sum_{k=1}^{T}\frac{1}{k}$. When $T \to \infty$, it holds that $\sum_{k=1}^{T}\frac{1}{k} \to \infty$ and $\varepsilon \to \infty$. Based on the above analysis, ε in this paper is smaller than that in [8] for some cases, which means the privacy performance can be enhanced. Moreover, it can be concluded from the expression of the index ε in Theorem 4.4 that the larger the parameter b of privacy noise is, the smaller the index ε is, thereby enhancing privacy protection. However, this improvement in privacy comes at the cost of reduced accuracy, as larger value of b inevitably yields larger value of variance and degrades accuracy.

Remark 4.6: This paper proposes an algorithm that blends estimation, control, and privacy to address the differentially private output consensus problem of the system (1). The main steps of this algorithm can be summarized as follows.
Estimation: Each agent designs the recursive projection operator (4) to estimate its neighbours' auxiliary variables.
Control: Based on its own system state and auxiliary variable, each agent constructs the impulsive controller (6) to achieve the mean-square output consensus with a convergence rate.
Privacy: By the sensitivity (25), each agent obtains the differential privacy index of the proposed algorithm as shown in Theorem 4.4.

V. Simulation

Consider the system (1) with parameters $A_i = \begin{pmatrix} 0 & 1 \\ a_i & e_i \end{pmatrix}$, $B_i = \begin{pmatrix} 0 & 0 \\ b_i & 0 \end{pmatrix}$, $C_i = (1\ 0)$, $\Theta_i = (1\ 0)^T$, $U_i = (-\frac{a_i}{b_i}\ 0)^T$, $a_i = -0.3 + 0.4i$, $b_i = -0.2 + 0.4i$, $e_i = -1.6 + 0.6i$, $i \in \mathcal{N}_1$, and $A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_i & e_i & f_i \end{pmatrix}$, $B_i = \begin{pmatrix} 0 & 0 & 0 \\ b_i & 0 & 0 \\ 0 & b_i & 0 \end{pmatrix}$, $C_i = (1\ 0\ 0)$, $\Theta_i = (1\ 0\ 0)^T$, $U_i = (0\ -\frac{a_i}{b_i}\ 0)^T$, $a_i = 1.3 - 0.3(i-5)$, $b_i = 2.2 - 0.2(i-5)$, $e_i = 3.4 - 0.4(i-5)$, $f_i = -0.1 - 0.3(i-5)$, $i \in \mathcal{N}_2$, where $\mathcal{N}_1 = \{1,2,3,4,5\}$ and $\mathcal{N}_2 = \{6,7,8,9,10\}$. The network topology is shown in Fig.1. For $(j,i) \in \mathcal{G}$, $h_{ij} = 1$.

In this case, the initial value of the system output is $y(0) = [10,20,30,40,50,60,70,80,90,100]^T$. Moreover, some system parameters are chosen as $\hat{x}_{ij}(0) = [9,19,19,29,39,39,49,49,59,59,69,69,79,89,89,99]^T$, $i = 1,\ldots,10$, $j \in N_i$, $M = 100$, $\vartheta_{ij} = 0$, $\nu_1 = 41$. The random noises obey the Laplacian distribution with each element satisfying $\omega_{j,l}(\cdot) \sim Lap(0,1)$.

To achieve the control target, the impulsive controller (6) for each agent is respectively designed with the control gain matrices $K_{1,1} = \begin{pmatrix} -10 & -0.8 \\ -0.9 & -0.1 \end{pmatrix}$, $K_{1,2} = \begin{pmatrix} -2 & 0.1 \\ 0.1 & 0.1 \end{pmatrix}$, $K_{1,3} = \begin{pmatrix} -2 & 0.12 \\ 0.1 & -0.1 \end{pmatrix}$, $K_{1,4} = \begin{pmatrix} -1.2 & 0.1 \\ 0 & -0.12 \end{pmatrix}$, $K_{1,5} = \begin{pmatrix} -1 & 0.1 \\ 0 & -0.9 \end{pmatrix}$, $K_{1,6} = \begin{pmatrix} -1 & -0.1 & -0.01 \\ -0.1 & 0 & 0 \\ 0 & -0.1 & -0.9 \end{pmatrix}$, $K_{1,7} = \begin{pmatrix} -1 & -0.1 & -0.1 \\ -0.1 & 0 & 0 \\ 0 & -0.1 & -0.9 \end{pmatrix}$, $K_{1,8} = \begin{pmatrix} -1.5 & -0.1 & -0.1 \\ -0.1 & 0 & 0 \\ 0 & -0.1 & -0.8 \end{pmatrix}$, $K_{1,9} = \begin{pmatrix} -1.8 & -0.1 & -0.1 \\ -0.1 & 0 & 0 \\ 0 & -0.3 & -0.9 \end{pmatrix}$, $K_{1,10} = \begin{pmatrix} -1.9 & -0.11 & -0.11 \\ -0.1 & 0 & 0 \\ 0 & -0.1 & -0.7 \end{pmatrix}$, $K_{2,1} = (5.7578\ 0)^T$, $K_{2,2} = (0.1428\ 0)^T$, $K_{2,3} = (0.7745\ 0)^T$, $K_{2,4} = (0.2727\ 0)^T$, $K_{2,5} = (0.2230\ 0)^T$, $K_{2,6} = (0.6111\ -0.4526\ 0)^T$, $K_{2,7} = (0.6517\ -0.4958\ 0)^T$, $K_{2,8} = (1.0791\ -0.5505\ 0)^T$, $K_{2,9} = (1.2867\ -0.6215\ 0)^T$, $K_{2,10} = (1.2845\ -0.7168\ 0)^T$. The time sequence of information transmission $\{\tau_k\}$ satisfies $\tau_k = 0.1k$. With the recursive projection operator (4) and the impulsive controller (6), the differentially private output consensus of the system (1) can be achieved. Fig.2 describes the simulation results of the system outputs. Fig.3 shows the simulation results of the auxiliary variables $\xi_i(\tau_k)$. Fig.4 shows the estimation results. Fig.5 shows the simulation result of $V_1(k)$ with the algorithms in this paper and in [14], which demonstrates that the convergence speed of $V_1(k)$ in this paper is faster than that in [14].

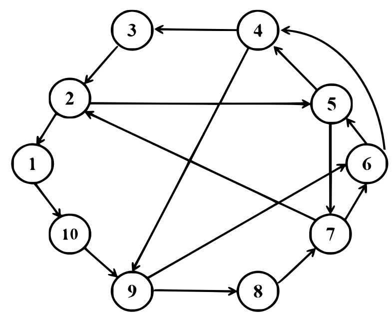

Fig. 1. Network topology.

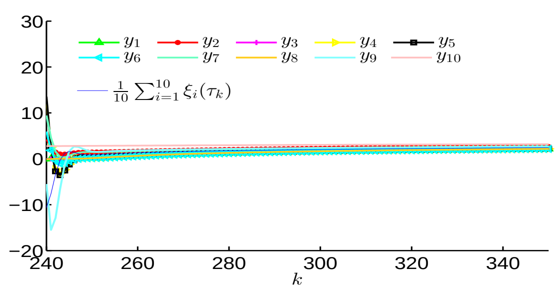

Fig. 2. Trajectories of system outputs $y_i(\tau_k)$ and $\frac{1}{10}\sum_{i=1}^{10}\xi_i(\tau_k)$.

VI. Conclusion

In this paper, the differentially private output consensus problem of continuous-time heterogeneous MASs under binary-valued communications has been addressed. By constructing a recursive projection algorithm and designing an impulsive controller, the mean-square output consensus with a convergence rate has been achieved. Moreover, the differential privacy index of the proposed mechanism has been obtained.

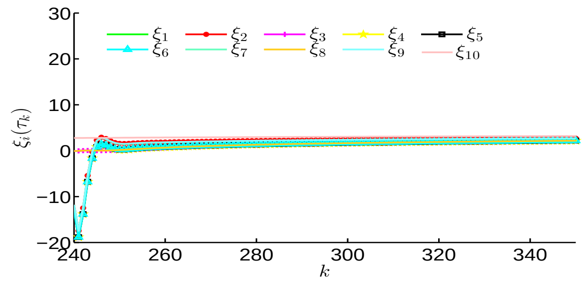

Fig. 3. Trajectories of auxiliary variables $\xi_i(\tau_k)$.

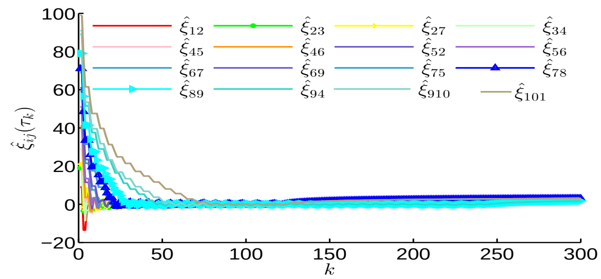

Fig. 4. Trajectories of estimations $\hat{\xi}_{ij}(\tau_k)$.

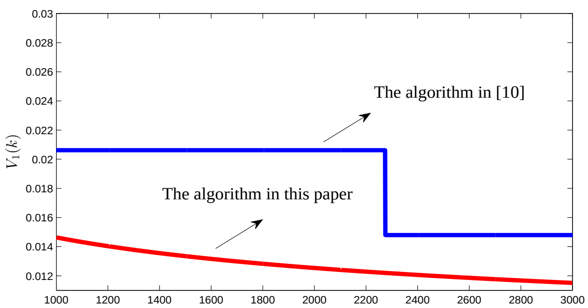

Fig. 5. Trajectory of $V_1(k)$.

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